Q&A

Non-Hermitian Hamiltonian of the system

$$H_0 = \begin{pmatrix} \omega_0 + \mathrm{i}\,[g_1\,|A_1|^2 - \gamma/2] & \mathrm{i}\kappa \\ \mathrm{i}\kappa & \omega_0 + \mathrm{i}\,[g_2\,|A_2|^2 - \gamma/2] \end{pmatrix}$$

This is the non-Hermitian Hamiltonian governs the system. A1 and A2 represent the photon-number-normalized amplitudes of the CCW and CW components of the pump modes, respectively, ω0 is the unpumped frequency of the Stokes cavity mode and γ is the cavity damping rate. $g_j = g_0/[1 + (2\mathrm{i}\Delta\Omega_j/\Gamma)]$ for j = 1, 2, is the Brillouin gain factor, where g0 is the gain coefficient, Γ is the gain bandwidth and $\Delta\Omega_j = \omega_{\mathrm{p}j} - \omega_{\mathrm{s}} - \Omega_{\mathrm{phonon}}$ is the frequency mismatch, ωpj is the pump photon frequency with ωs the Stokes lasing frequency (an eigenfrequency of Hamiltonian above).

The real part of the Brillouin gain factor leads to amplification of the Stokes mode, whereas the imaginary part is responsible for dispersion and consequently mode pulling. κ is the dissipative coupling rate between the two SBL modes.

In a standing-wave mode basis, the optical loss induced by the fiber taper or any other spatially localized absorption or dissipative scattering element will be different for each mode and can be captured by the following contribution to the Hamiltonian:

$$H_{\mathrm{taper}} = \begin{pmatrix} -i\gamma_1 & 0 \\ 0 & -i\gamma_2 \end{pmatrix}$$

Changing to a traveling wave basis (CW and CCW) by using the relation $|\Phi_\pm\rangle = (|\mathrm{CW}\rangle \pm |\mathrm{CCW}\rangle / \sqrt{2}$ gives the following Hamiltonian in the new basis,

$$H_{\mathrm{taper}} = \begin{pmatrix} -i\gamma_{\mathrm{common}} & 0 \\ 0 & -i\gamma_{\mathrm{common}} \end{pmatrix} + \begin{pmatrix} 0 & i\kappa \\ i\kappa & 0 \end{pmatrix}$$

where $\gamma_{\mathrm{common}} = (\gamma_1 + \gamma_2)/2$ and $\kappa = (\gamma_2 - \gamma_1)/2$. The first term is the common loss (out-coupling loss of the taper) while the second term is the dissipative backscattering. Most of the dissipative backscattering originates from taper scattering. Specifically, changes in the contact position of the taper on the resonator are observed to vary κ.

The steady-state lasing condition requires the power loss rate γ to be balanced by the Brillouin gain, which leads to the clamping condition of the pump powers,

$$|A_j|^2 = \gamma \frac{1 + (2\Delta\Omega_j / \Gamma)^2}{2g_0} \quad .$$

As a result, above the lasing threshold Hamiltonian is simplified to the following form:

$$H_0 = \begin{pmatrix} \omega_0 + \frac{\gamma}{\Gamma}\Delta\Omega_1 & i\kappa \\ i\kappa & \omega_0 + \frac{\gamma}{\Gamma}\Delta\Omega_2 \end{pmatrix}$$

With the introduction of κ, the lasing system exhibits a frequency locking–unlocking transition when varying the pump detuning frequency. The locking regime is known to create a sensing dead band for rotations in ring laser gyroscopes. In the frequency-unlocked regime, the two lasing modes have distinct angular frequencies ω_{s+} and ω_{s-}, which are the eigenvalues of the Hamiltonian.

$$\omega_{s\pm} - \omega_r = \frac{\gamma/(2\Gamma)}{1+\gamma/\Gamma}\left(\Delta\omega_p \pm \sqrt{\Delta\omega_p^2 - \Delta\omega_c^2}\right)$$

where $\omega_r = \{\omega_0 + [\gamma(\omega_{p1} - \Omega_{phonon})/\Gamma]\}/[1+\gamma/\Gamma]$, $\Delta\omega_p = \omega_{p2} - \omega_{p1}$ is the pump detuning frequency and $\Delta\omega_c = 2\Gamma\kappa/\gamma$ is the critical pump frequency detuning at which the system state is at an EP.

Langevin Formalism

For readability, all cw subscript will be replaced by $\bar{1}$ and all ccw subscript will be replaced by $\bar{2}$. The modes are pumped at angular frequencies $\omega_{\mathrm{P},\bar{1}}$ and $\omega_{\mathrm{P},\bar{2}}$. The loss rate of phonon modes is denoted as Γ (also known as the gain bandwidth) and the loss rate of the SBL modes are assumed equal and denoted as γ. In addition, coupling between the two SBL modes is separated as a dissipative part and conservative part, denoted as κ and χ, respectively. These rates will be assumed to satisfy $\Gamma \gg \gamma \gg |\kappa|$ to simplify the calculations, which is a posteriori verified in our system. In the following analysis, we will treat the SBL modes and phonon modes quantum mechanically and define $a_{\bar{1}}$ $(a_{\bar{2}})$ and $b_{\bar{1}}$ $(b_{\bar{2}})$ as the lowering operators of the cw (ccw) components of the SBL and phonon modes, respectively. Meanwhile, pump modes are treated as a noise-free classical fields A1 and A2 (photon-number-normalized amplitudes).

Using these definitions, the full equations of motion for the SBL and phonon modes read

$$\dot{a}_{\bar{1}} = -\left(\frac{\gamma}{2} + i\omega + i\delta\omega_{\bar{1}}\right) a_{\bar{1}} + (\kappa + i\chi) a_{\bar{2}} - i g_{ab} A_{\bar{2}} b_{\bar{2}}^{\dagger} \exp(-i\omega_{\mathrm{P},\bar{2}} t) + F_{\bar{1}}(t)$$

$$\dot{a}_{\bar{2}} = -\left(\frac{\gamma}{2} + i\omega + i\delta\omega_{\bar{2}}\right) a_{\bar{2}} + (\kappa^* + i\chi^*) a_{\bar{1}} - i g_{ab} A_{\bar{1}} b_{\bar{1}}^{\dagger} \exp(-i\omega_{\mathrm{P},\bar{1}} t) + F_{\bar{2}}(t)$$

$$\dot{b}_{\bar{1}}^{\dagger} = -\left(\frac{\Gamma}{2} - i\Omega_{\mathrm{phonon}}\right) b_{\bar{1}}^{\dagger} + i g_{ab} A_{\bar{1}}^* a_{\bar{2}} \exp(i\omega_{\mathrm{P},\bar{1}} t) + f_{\bar{1}}^{\dagger}(t)$$

$$\dot{b}_{\bar{2}}^{\dagger} = -\left(\frac{\Gamma}{2} - i\Omega_{\mathrm{phonon}}\right) b_{\bar{2}}^{\dagger} + i g_{ab} A_{\bar{2}}^* a_{\bar{1}} \exp(i\omega_{\mathrm{P},\bar{2}} t) + f_{\bar{2}}^{\dagger}(t)$$

where g_{ab} is the single-particle Brillouin coupling coefficient. F(t) and f(t) are the fluctuation operators.

Two SBLs

For two pairs of photon and phonon modes with coupling on the optical modes. We write the equations of motion for the SBL modes:

$$\frac{d}{dt}\overline{\alpha}_{\overline{1}} = -i(\omega_{\mathrm{S},\overline{1}} - \omega)\overline{\alpha}_{\overline{1}} + \frac{\Gamma}{\gamma+\Gamma}(\kappa + i\chi)\alpha_{\overline{2}} + \overline{F}_{\overline{1}}(t)$$

$$\frac{d}{dt}\overline{\alpha}_{\overline{2}} = -i(\omega_{\mathrm{S},\overline{2}} - \omega)\overline{\alpha}_{\overline{2}} + \frac{\Gamma}{\gamma+\Gamma}(\kappa^* + i\chi^*)\alpha_{\overline{1}} + \overline{F}_{\overline{2}}(t)$$

No additional coupling occurs between the other components of the SBL eigenstates $\overline{\alpha}_{\overline{1}}$ and $\overline{\alpha}_{\overline{2}}$. Thus we can approximate the optical mode $\alpha_{\overline{1}}$ with the composite SBL mode $\overline{\alpha}_{\overline{1}}$. The equations now become

$$\frac{d}{dt}\overline{\alpha}_{\overline{1}} = -i(\omega_{\mathrm{S},\overline{1}} - \omega)\overline{\alpha}_{\overline{1}} + (\overline{\kappa} + i\overline{\chi})\overline{\alpha}_{\overline{2}} + \overline{F}_{\overline{1}}(t)$$

$$\frac{d}{dt}\overline{\alpha}_{\overline{2}} = -i(\omega_{\mathrm{S},\overline{2}} - \omega)\overline{\alpha}_{\overline{2}} + (\overline{\kappa}^* + i\overline{\chi}^*)\overline{\alpha}_{\overline{1}} + \overline{F}_{\overline{2}}(t)$$

where we have defined mode-pulled coupling rates $\overline{\kappa} = \kappa\Gamma/(\gamma+\Gamma)$ and $\overline{\chi} = \chi\Gamma/(\gamma+\Gamma)$.

We can write $\overline{\alpha}_j(t) = \sqrt{N_j}\exp(-i\phi_j)$ with j = 1, 2, and ignore amplitude fluctuations (Amplitude fluctuations have been ignored on account of quenching of these fluctuations above laser threshold).

The equations of motion for the phases are,

$$\frac{d}{dt}\phi_{\bar{1}} = (\omega_{\mathrm{S},\bar{1}} - \omega) - q\mathrm{Im}[(\bar{\kappa} + i\bar{\chi})\mathrm{e}^{(i\phi_{\bar{1}} - i\phi_{\bar{2}})}] + \frac{i}{2\sqrt{N_{\bar{1}}}}(\overline{F}_{\bar{1}}(t)\mathrm{e}^{i\phi_{\bar{1}}} - \overline{F}^{\dagger}_{\bar{1}}(t)\mathrm{e}^{-i\phi_{\bar{1}}})$$

$$\frac{d}{dt}\phi_{\bar{2}} = (\omega_{\mathrm{S},\bar{2}} - \omega) - q^{-1}\mathrm{Im}[(\bar{\kappa}^* + i\bar{\chi}^*)\mathrm{e}^{(i\phi_{\bar{2}} - i\phi_{\bar{1}})}] + \frac{i}{2\sqrt{N_{\bar{2}}}}(\overline{F}_{\bar{2}}(t)\mathrm{e}^{i\phi_{\bar{2}}} - \overline{F}^{\dagger}_{\bar{2}}(t)\mathrm{e}^{-i\phi_{\bar{2}}})$$

where we have defined the amplitude ratio $q = \sqrt{N_{\bar{2}}/N_{\bar{1}}}$ for simplicity. As we measure the beatnote frequency, it is convenient to define $\phi \equiv \phi_{\bar{2}} - \phi_{\bar{1}}$ from which we obtain,

$$\frac{d\phi}{dt} = (\omega_{\mathrm{S},\bar{2}} - \omega_{\mathrm{S},\bar{1}}) + \mathrm{Im}\left\{\left[q(\bar{\kappa} + i\bar{\chi}) + q^{-1}(\bar{\kappa} - i\bar{\chi})\right]\mathrm{e}^{-i\phi}\right\} + \Phi(t)$$

Where the combined noise term and its correlation are given by

$$\Phi = -\frac{i}{2\sqrt{N_{\bar{1}}}}(\overline{F}_{\bar{1}}(t)\mathrm{e}^{i\phi_{\bar{1}}} - \overline{F}^{\dagger}_{\bar{1}}(t)\mathrm{e}^{-i\phi_{\bar{1}}}) + \frac{i}{2\sqrt{N_{\bar{2}}}}(\overline{F}_{\bar{2}}(t)\mathrm{e}^{i\phi_{\bar{2}}} - \overline{F}^{\dagger}_{\bar{2}}(t)\mathrm{e}^{-i\phi_{\bar{2}}})$$

$$\langle\Phi(t)\Phi(t')\rangle = \left(\frac{\Gamma}{\gamma+\Gamma}\right)^2\left[\left(\frac{1}{2N_{\bar{1}}} + \frac{1}{2N_{\bar{2}}}\right)\gamma(N_{\mathrm{th}} + n_{\mathrm{th}} + 1) + \frac{2}{\sqrt{N_{\bar{1}}N_{\bar{2}}}}\left(N_{\mathrm{th}} + \frac{1}{2}\right)\mathrm{Re}(\kappa\mathrm{e}^{-i\phi(t)})\right]\delta(t-t')$$

Obtain the linewidth from Adler equation

We define $z_\phi = \exp(-i\bar{\phi})$ and rewrite

$$\frac{d}{dt} z_\phi = -i z_\phi \left(\Delta\omega_{\rm D} + \Delta\omega_{\rm EP} \frac{z_\phi - z_\phi^{-1}}{2i} + \Phi\right)$$

The solution to the Adler equation is periodic when no noise is present. To see this explicitly we use a linear fractional transform:

$$z_t = \frac{(\Delta\omega_{\rm D} - \Delta\omega_{\rm S}) z_\phi + i\Delta\omega_{\rm EP}}{\Delta\omega_{\rm EP} z_\phi + i(\Delta\omega_{\rm D} - \Delta\omega_{\rm S})}, \quad z_\phi = -i \frac{(\Delta\omega_{\rm D} - \Delta\omega_{\rm S}) z_t - \Delta\omega_{\rm EP}}{\Delta\omega_{\rm EP} z_t - (\Delta\omega_{\rm D} - \Delta\omega_{\rm S})}, \quad |z_\phi| = |z_t| = 1$$

$$\frac{1}{z_t} \frac{d}{dt} z_t = i\Delta\omega_{\rm S} - i \frac{\Delta\omega_{\rm EP}(z_t + z_t^{-1})/2 - \Delta\omega_{\rm D}}{\Delta\omega_{\rm S}} \Phi$$

The linewidth can be found from the spectral density, which is given by the Fourier transform of the correlation function:

$$W_E(\omega) \propto \mathcal{F}_\tau\{\langle z_\phi^*(t) z_\phi(t+\tau)\rangle\}(\omega)$$

By applying the Fokker-Planck equation and calculation, the linewidth of the fundamental frequency can be found through

$$W_{E,1}(\omega) \propto \frac{\Delta\overline{\omega}_{\mathrm{FWHM}}}{(\omega - \Delta\omega_{\mathrm{S}})^2 + \Delta\overline{\omega}_{\mathrm{FWHM}}^2/4}$$

with

$$\Delta\overline{\omega}_{\mathrm{FWHM}} = \frac{\Delta\omega_{\mathrm{D}}^2 + \Delta\omega_{\mathrm{EP}}^2/2}{\Delta\omega_{\mathrm{S}}^2}\Delta\omega_{\mathrm{FWHM}} = \frac{\Delta\omega_{\mathrm{D}}^2 + \Delta\omega_{\mathrm{EP}}^2/2}{\Delta\omega_{\mathrm{D}}^2 - \Delta\omega_{\mathrm{EP}}^2}\Delta\omega_{\mathrm{FWHM}}$$

$$\Delta\omega_{\mathrm{S}} = \mathrm{sgn}(\Delta\omega_{\mathrm{D}})\sqrt{\Delta\omega_{\mathrm{D}}^2 - \Delta\omega_{\mathrm{EP}}^2}$$

$$\Delta\omega_{\mathrm{D}} = \frac{\gamma}{\Gamma + \gamma}\Delta\omega_{\mathrm{P}} + \frac{\Gamma}{\Gamma + \gamma}\Delta\omega_{\mathrm{Kerr}}$$

$$\Delta\omega_{\mathrm{P}} = \omega_{\mathrm{P},\overline{1}} - \omega_{\mathrm{P},\overline{2}}$$

$$\Delta\omega_{\mathrm{Kerr}} = \eta(N_{\overline{2}} - N_{\overline{1}}) = \frac{\eta\Delta P_{\mathrm{SBL}}}{\gamma_{\mathrm{ex}}\hbar\omega}$$